Unit- IV Jacobian, Error & Maxima - Minima.

25|3|21

Topic Consists of

1) Jacobian
2) Error & Approx
3) Maxima & Minima.

Jacobian

Define Suppose u & v are Fn of x & y given by $u = f(x,y)$, $v = g(x,y)$. Jacobian of u, v wrt. x, y is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or J & given by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Type- I Example on Define

1) If $u = x^2 - y^2$, $v = 2xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$

→ Given $u = x^2 - y^2$, $v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4x^2 + 4y^2 = \boxed{4(x^2+y^2)}$$

2) If $x = r\cos\theta$, $y = r\sin\theta$

Find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

$\longrightarrow$ Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\boxed{= r}$$

3) If $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(r,\theta)}{\partial(x,y)}$?

- x, y (Cartesian Co-ordi) $\longrightarrow$ r, θ (Polar Co-ord) } 2 dim

- 3-dim
  1) Cartesian — (x, y, z)
  2) Spherical Polar (r, θ, ϕ)
  3) Cylindrical Co-ord (ρ, ϕ, z).

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3) Consider rel^n bet^n cartesian & spherical polar co-ordinates as $x = r\sin\theta\cos\phi$, ~~y = rs~~ $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

→ Given $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking r & $r\sin\theta$ common from C_2 & C_3.

$$= r \times r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \Big[\sin\theta\cos\phi\{0 + \sin\theta\cos\phi\}$$
$$- \cos\theta\cos\phi\{0 - \cos\theta\cos\phi\}$$
$$- \sin\phi\{-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\} \Big]$$

$$= r^2 \sin\theta \left[\sin^2\theta \cos^2\phi + \cos^2\theta \cos^2\phi + \sin^2\phi \left\{ \sin^2\theta + \cos^2\theta \right\} \right]$$

$$= r^2 \sin\theta \left[\cos^2\phi \left\{ \underbrace{\sin^2\theta + \cos^2\theta}_{1} \right\} + \sin^2\phi \{1\} \right]$$

$$= r^2 \sin\theta \left[\cos^2\phi + \sin^2\phi \right]$$

$$\boxed{= r^2 \sin\theta}$$

Type-II Jacobian of Composite Function

I) Suppose u & v are fun of x, y & x, y are funs of r, θ given by

$u = f_1(x, y)$ & $x = g_1(r, \theta)$

$v = f_2(x, y)$ $y = g_2(r, \theta)$

$$u, v \longrightarrow x, y \longrightarrow r, \theta .$$

$$\boxed{\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}}$$

II) Suppose $u = f_1(x, y)$, $v = f_2(x, y)$

$$u, v \longrightarrow x, y .$$

Let $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ & $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$ then $\boxed{J_1 \times J_2 = 1}$

II A Examples on Jacobian of Composite fn

1) If $u = 2xy$, $v = x^2 - y^2$ & $x = e^{r}\sec\theta$, $y = e^{r}\tan\theta$

Find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

$\longrightarrow$ Given $u = 2xy$, $v = x^2 - y^2$ & $x = e^{r}\sec\theta$, $y = e^{r}\tan\theta$.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(r,\theta)} \qquad u,v \longrightarrow x,y \longrightarrow r,\theta .$$

$$J_1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2 = \boxed{-4(x^2+y^2)}$$

$$J_2 = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} e^{r}\sec\theta & e^{r}\sec\theta\tan\theta \\ e^{r}\tan\theta & e^{r}\sec^2\theta \end{vmatrix}$$

$$= e^{2r}\sec^3\theta - e^{2r}\sec\theta\tan^2\theta = e^{2r}\sec\theta\left[\sec^2\theta - \tan^2\theta\right]$$

$$\boxed{J_2 = e^{2r}\sec\theta}$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4(x^2+y^2)\times e^{2r}\sec\theta = \boxed{-4e^{2r}\sec\theta\,(x^2+y^2)}$$

2) If $u = ax+by$, $v = bx - ay$ & $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

→ Given $u = ax+by$ & $x = r\cos\theta$
$v = bx - ay$ & $y = r\sin\theta$

$$u, v \longrightarrow x, y \longrightarrow r, \theta$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= J_1 \times J_2 \quad \text{—} \quad (1)$$

$$J_1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} a & b \\ b & -a \end{vmatrix} = \boxed{-a^2 - b^2}$$

$$J_2 = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

from eqⁿ (1)

$$\frac{\partial(u,v)}{\partial(r,\theta)} = J_1 \times J_2 = \boxed{-(a^2+b^2)r}$$

25/3/21 (1)

Type II B) Use of $JJ' = 1$ to find jacobian

1) If $x = e^u \sec v$, $y = e^u \tan v$, Find $\frac{\partial(u,v)}{\partial(x,y)}$

→ Note $\left\{ \begin{array}{l} \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \text{example can be solved by define} \\ x \to u,v \\ y \to u,v \end{array} \right.$

→ Given $x = e^u \sec v$, $y = e^u \tan v$

$$x, y \longrightarrow u, v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^u \sec v & e^u \sec v \tan v \\ e^u \tan v & e^u \sec^2 v \end{vmatrix}$$

$$= e^{2u} \sec^3 v - e^{2u} \tan^2 v \sec v$$

$$= e^{2u} \sec v \, [\underbrace{\sec^2 v - \tan^2 v}_{1}]$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = e^{2u} \sec v = J}$$

But $J' = \frac{\partial(u,v)}{\partial(x,y)}$ & $JJ' = 1$

$$J' = \frac{1}{J}$$

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{e^{2v} \sec v} = e^{-2u} \cos v}$$

2) If $u = x - xy$, $v = xy$, find $\dfrac{\partial(x,y)}{\partial(u,v)}$

$\longrightarrow$ Given $u = x - xy$, $v = xy$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x - xy + xy = \boxed{x}$$

But $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ & $JJ' = 1$

$$\Rightarrow J' = \frac{1}{J}$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{x}}$$

3) If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$

Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

$\longrightarrow$ Given $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 0 - 2v\left[0 - 4uw\right] + 2w\left[4uv - 0\right]$$

$$= 8uvw + 8uvw$$

$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = 16uvw = J}$$

But $\quad JJ' = 1 \quad$ where $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

$$J' = \frac{1}{J}$$

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{16uvw}}$$

Jacobian of ~~for~~ Implicit Function

I) Suppose u, v, w are implicit Fu^n of x, y, z.

given by
$$f_1(u, v, w, x, y, z) = 0$$
$$f_2(u, v, w, x, y, z) = 0$$
$$f_3(u, v, w, x, y, z) = 0$$

then
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

$$= (-1)^3 \frac{\partial(f_1, \cancel{f_2, f_3})}{\partial(x, y, z)} \times \frac{\partial(u, v, w)}{\partial(f_1, \cancel{f_2, f_3})}$$

II) Suppose u, v are implicit Fu^n of x, y

given by
$$f_1(u, v, x, y) = 0$$
$$f_2(u, v, x, y) = 0$$

then
$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Explicit Fu^n / Implicit Fu^n

A Fu^n $f(x, y) = 0$ is said to be explicit if x & y are seperable else it is implicit Fu^n.

1) $y = x^2 + 3x - 1 \longrightarrow$ explicit

2) $x^2y + y\sin x = 17x^2 + 3x - 1$

$\Rightarrow y(x^2 + \sin x) = \text{— '' —}$

$$y = \frac{17x^2 + 3x - 1}{x^2 + \sin x} \Rightarrow \text{explicit}$$

3) $x^3y^2 + x^2y + y^2 = 13 \rightarrow$ implicit fun

$\Rightarrow$ Not possible to separate x & y.

4) $u^2 \cancel{+} v^2 + x^2 + y^2 = 0$, $uv + xy = 0$

$\longrightarrow$ This is explicit if $u = f_1(x,y)$, $v = f_2(x,y)$ which is very difficult so this is implicit fun.

5) $u = x + y + z$, $v = xy + yz + zx$, $w = xyz$

$\longrightarrow$ $u, v, w \longrightarrow x, y, z$.

u, v, w are explicit fun of x, y, z.

6) $u + v + w = x + y + z$, $\underline{u}v + vw + wu = \underline{x}^2 + y^2 + z^2$

$\underline{u}vw = \underline{x}^3 + y^3 + z^3$

$\longrightarrow$ Implicit fun.

$\left.\begin{array}{l} u \rightarrow x, y, z \\ v \rightarrow x, y, z \\ w \rightarrow x, y, z \end{array}\right\}$ explicit

2) ~~Type of~~ III Examples on Jacobian of Implicit Fu^n

1) If $x^2+y^2+u^2+v^2=0$, $xy+uv=0$, find $\frac{\partial(u,v)}{\partial(x,y)}$

$\longrightarrow$ Given u,v are implicit Fu^n of x,y given by

$f_1 = u^2+v^2+x^2+y^2$

$f_2 = uv+xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1 f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{N}{D} \quad — ①$$

$$N = \frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2-2y^2 = \boxed{2(x^2-y^2)}$$

$$D = \frac{\partial(f_1 f_2)}{\partial(u,v)} = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ v & u \end{vmatrix}$$

$$= 2u^2-2v^2 = \boxed{2(u^2-v^2)}$$

$$① \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D} = \frac{2(x^2-y^2)}{2(u^2-v^2)} = \boxed{\frac{x^2-y^2}{u^2-v^2}}$$

2) If $x+y+z = u$, $\overset{y+z}{\cancel{x+y}} = uv$, $\overset{z}{\cancel{x}} = uvw$

Find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

→ Note example can be solved by

1) $\boxed{\text{Define}} \Rightarrow$ $x, y, z \to u, v, w$ — Explicit Fuⁿ Method

$z = uvw$, $y = uv - z$, $x = u - y - z$

$\boxed{y = uv - uvw}$

$= u - \{uv - uvw) - uvw$

✓ $\boxed{x = u - uv}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right]$$

$$+ u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$\boxed{= u^2v}$$

2) Implicit Fun Method

Let $f_1 = x+y+z-u$

$f_2 = y+z-uv$

$f_3 = z-uvw$

Determinant of diag, upper triang & lower triang matrix is product of diagonal elements.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)}} = -\frac{N}{D} \quad \text{--- (1)}$$

- $$N = \frac{\partial(f_1, f_2 f_3)}{\partial(u,v,w)} = \begin{vmatrix} f_{1u} & f_{1v} & f_{1w} \\ f_{2u} & f_{2v} & f_{2w} \\ f_{3u} & f_{3v} & f_{3w} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = (-1)(-u)(-uv) = \boxed{-u^2v}$$

Lower triang Matrix

- $$D = \frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)} = \begin{vmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) = \boxed{1}$$

Upper triangula matrix

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-N}{D} = -\frac{(-u^2v)}{1} = \boxed{u^2v}$$

3) If $u+v+w = x+y+z$, $uv+vw+wu = x^2+y^2+z^2$

& $uvw = \frac{1}{3}(x^3+y^3+z^3)$ then prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

$\longrightarrow$ Given u, v, w are implicit fun of x, y, z.

given by $f_1 = u+v+w - (x+y+z)$

$f_2 = uv+vw+wu - (x^2+y^2+z^2)$

$f_3 = uvw - \frac{1}{3}(x^3+y^3+z^3)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{N}{D} \quad \text{--- (1)}$$

- $N = \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{vmatrix}$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

Taking $-1, -2$ & -1 common from R_1, R_2 & R_3 respect..

$$N = (-1)(-2)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$C_1 \to C_1 - C_2 \quad , \quad C_2 \to C_2 - C_3$

$$N = -2\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

$$= -2(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$

$$= -2(x-y)(y-z)\left[0+0+1\,\{(y+z)-(x+y)\}\right]$$

$$\boxed{N = -2(x-y)(y-z)(z-x)}$$

- $$D = \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} f_{1u} & f_{1v} & f_{1w} \\ f_{2u} & f_{2v} & f_{2w} \\ f_{3u} & f_{3v} & f_{3w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$C_1 \to C_1 - C_2, \quad C_2 \to C_2 - C_3.$

$$D = \begin{vmatrix} 0 & 0 & 1 \\ v-u & w-v & u+v \\ vw-uw & uw-uv & uv \end{vmatrix}$$

$$= (v-u)(w-v)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & u+v \\ w & u & uv \end{vmatrix}$$

$$= \left\{\underline{-}\ (u-v)\right\}\left\{\underline{-}\ (v-w)\right\}\left[0+0+1\left\{u-w\right\}\right]$$

$$= (u-v)(v-w)\underset{\substack{\Downarrow \\ \text{taking } -1 \text{ common}}}{(u-w)}$$

$$\boxed{D = -(u-v)(v-w)(w-u)}$$

$$① \Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{N}{D} = -\frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}}$$

1) $u+v+w = x+y+z$

$uv+vw+wu = x^2+y^2+z^2$

$uvw = \frac{1}{3}(x^3+y^3+z^3)$

<u>Explicit</u>

$u \to x, y, z$ | or | $x \to u, v, w$
$v \to x, y, z$ | | $y \to u, v, w$
$w \to x, y, z$ | | $z \to u, v, w$

which is difficult

$\Rightarrow$ Fuⁿ is implicit.

4) Partial derivative of implicit Fuⁿ

1) Given $u = x^2+y^2+z^2$, $v = xy+yz+zx$

$w = xyz$

$\to$ a) Find $\frac{\partial u}{\partial x}$

$u \to$ depend

$x \to$ independent

$u \longrightarrow x$

$\frac{du}{dx}$

$\checkmark\ u \to x, y, z.$

① $\Rightarrow$ $u = x^2+y^2+z^2$

$\frac{\partial u}{\partial x} = \cancel{0} + \cancel{2y}\ \ 2x+0+0 = 2x$

b) $\frac{\partial v}{\partial y} = x+z+0 = x+z$ c) $\frac{\partial w}{\partial x} = yz$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$

d) $\frac{\partial x}{\partial u}$ $x \to u, v, w$ <u>* x implicit Fuⁿ of u, v, w</u>

$u = x^2 + y^2 + z^2$, $\quad v = xy + yz + zx$, $\quad w = xyz$

u, v, w are explicit fun of x, y, z

$u \to x, y, z$, $\quad v \to x, y, z$, $\quad w \to x, y, z$

$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial y}$ ✓ → simple

x, y, z are <u>implicit</u> fun of u, v, w

<u>explicit</u> $x \to u, v, w$, $\quad y \to u, v, w$, $\quad z \to u, v, w$

To find $\frac{\partial x}{\partial u}$ → Use Jacobian method.

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17 u, v, w are implicit fuⁿs of x, y, z given by

$f_1(u, v, w, x, y, z) = 0$

$f_2(u, v, w, x, y, z) = 0$

$f_3(u, v, w, x, y, z) = 0$ then

$$\frac{\partial u}{\partial x} = - \frac{\dfrac{\partial (f_1, f_2, f_3)}{\partial (x, v, w)}}{\dfrac{\partial (f_1, f_2, f_3)}{\partial (\underset{\downarrow \atop x}{u}, v, w)}}$$

$$\boxed{\frac{\partial (f_1, \cancel{f_2\, f_3})}{\partial (x, \cancel{v, w})} \times \frac{\partial (u, \cancel{v, w})}{\partial (f_1, \cancel{f_2, f_3})}}$$

$$\frac{\partial v}{\partial y} =$$

$$\frac{\partial v}{\partial y} = - \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, w)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, \underset{\substack{\downarrow \\ y}}{v}, w)}} \quad , \quad \frac{\partial w}{\partial x} = - \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, x)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, \underset{\substack{\downarrow \\ x}}{w})}}$$

$$\frac{\partial u}{\partial z} = - \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(z, v, w)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(\underset{\substack{\downarrow \\ z}}{u}, v, w)}} \quad , \quad \frac{\partial x}{\partial u} = - \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(\underset{\substack{\downarrow \\ u}}{x}, y, z)}}$$

II) Suppose u, v are implicit fn of x, y given by

$$f_1(u, v, x, y) = 0$$
$$f_2(u, v, x, y) = 0 \quad \text{then}$$

$$\frac{\partial u}{\partial x} = - \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(\underset{\substack{\downarrow \\ x}}{u}, v)}} \quad , \quad \frac{\partial v}{\partial y} = - \frac{\dfrac{\partial(f_1, f_2)}{\partial(u, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, \underset{\substack{\downarrow \\ y}}{v})}}$$

$$\frac{\partial x}{\partial v} = - \frac{\dfrac{\partial(f_1, f_2)}{\partial(v, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(\underset{\substack{\downarrow \\ v}}{x}, y)}}$$

1) If $x = u^2 - v^2$, $y = uv$ find $\frac{\partial u}{\partial x}$

→ Given u & v are implicit fun of x, y

$$f_1 = u^2 - v^2 - x$$

$$f_2 = uv - y$$

$$\frac{\partial u}{\partial x} = - \frac{\frac{\partial (f_1, f_2)}{\partial (x, v)}}{\frac{\partial (f_1, f_2)}{\partial (\underset{\substack{\downarrow \\ x}}{u}, v)}} = -\frac{N}{D} \quad - (1)$$

$$N = \frac{\partial (f_1, f_2)}{\partial (x, v)} = \begin{vmatrix} f_{1x} & f_{1v} \\ f_{2x} & f_{2v} \end{vmatrix} = \begin{vmatrix} -1 & -2v \\ 0 & u \end{vmatrix} = \boxed{-u}$$

$$D = \frac{\partial (f_1, f_2)}{\partial (\cancel{x} u, v)} = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

$$= 2u^2 + 2v^2 = \boxed{2(u^2 + v^2)}$$

(1) ⇒

$$\frac{\partial u}{\partial x} = -\frac{N}{D} = -\frac{-u}{2(u^2+v^2)}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{u}{2(u^2+v^2)}}$$

2) If $ux+vy=0$ & $\frac{u}{x}+\frac{v}{y}=1$, find $\left(\frac{\partial u}{\partial x}\right)_y$

→ Given u, v are implicit fn of x, y given by

$$f_1 = ux + vy$$

$$f_2 = \frac{u}{x} + \frac{v}{y} - 1$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} = -\frac{N}{D} \quad — (1)$$

x

- $N = \frac{\partial(f_1, f_2)}{\partial(x, v)} = \begin{vmatrix} f_{1x} & f_{1v} \\ f_{2x} & f_{2v} \end{vmatrix} = \begin{vmatrix} u & y \\ \frac{-u}{x^2} & \frac{1}{y} \end{vmatrix} = \boxed{\frac{u}{y} + \frac{uy}{x^2}} = \boxed{\frac{ux^2+uy^2}{x^2 y}}$

- $D = \frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} x & y \\ \frac{1}{x} & \frac{1}{y} \end{vmatrix}$

$$= \frac{x}{y} - \frac{y}{x} = \boxed{\frac{x^2-y^2}{xy}}$$

$$(1) \Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = -\frac{N}{D} = -\frac{\frac{u(x^2+y^2)}{x^2 y}}{\frac{x^2-y^2}{xy}}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{u(x^2+y^2)}{\cancel{x^2 y}_{\,x}} \times \frac{\cancel{xy}}{x^2-y^2} = \boxed{\frac{u(x^2+y^2)}{x(y^2-x^2)}}$$

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①

3) If $u = x+y+z$, $v = x^2+y^2+z^2$

$w = x^3+y^3+z^3$ p.t. $\frac{\partial x}{\partial u} = \frac{yz}{(x-y)(x-z)}$.

→ Given x, y, z are implicit fn of u, v, w given by

$$f_1 = x+y+z-u$$
$$f_2 = x^2+y^2+z^2-v$$
$$f_3 = x^3+y^3+z^3-w$$

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(\underset{\downarrow \atop u}{x}, y, z)}} = -\frac{N}{D} \quad ①$$

$$N = \frac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)} = \begin{vmatrix} f_{1u} & f_{1y} & f_{1z} \\ f_{2u} & f_{2y} & f_{2z} \\ f_{3u} & f_{3y} & f_{3z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2y & 2z \\ 0 & 3y^2 & 3z^2 \end{vmatrix}$$

$$= -1\{6yz^2 - 6y^2z\} - 1\{0\} + 1\{0\}$$

$$\boxed{N = -6yz(z-y)}$$

$$D = \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ 3(x^2-y^2) & 3(y^2-z^2) & 3z^2 \end{vmatrix}$$

$$= \cancel{2}(x-y)\cdot(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 2 & 2 & 2z \\ 3(x+y) & 3(y+z) & 3z^2 \end{vmatrix}$$

$$= (x-y)(y-z)\left[0+0+1\left\{6(y+z) - 6(x+y)\right\}\right]$$

$$\boxed{D = (x-y)(y-z)\cdot 6(z-x)}$$

① $\Rightarrow$ $$\frac{\partial x}{\partial u} = -\frac{N}{D} = + \frac{+\cancel{6}yz(z-y)}{\cancel{6}(x-y)\underline{(y-z)}\underline{(z-x)}}$$

$$\frac{\partial x}{\partial u} = \frac{yz\cancel{(z-y)}}{(x-y)\left\{+\cancel{(z-y)}\right\}\left\{+(x-z)\right\}} = \boxed{\frac{yz}{(x-y)(x-z)}}$$

5) Functional Dependance

Suppose $u = f_1(x,y)$, $v = f_2(x,y)$.

u. & v are said to be functionally dependent if $u = F(v)$ or $v = F(u)$.

e.g. 1) $u = x+y$, $v = \sin(x+y)$

$\longrightarrow$ Given $v = \sin(x+y)$

$$\boxed{v = \sin u} \checkmark \quad \text{Rel}^n$$

$\Rightarrow$ u, v are fu^n dependent.

2) $u = \tan^{-1}x + \tan^{-1}y$, $v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

$\longrightarrow$ Using $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

Given $u = \tan^{-1}x + \tan^{-1}y$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right) = v \Rightarrow \boxed{u = v} \text{ Rel}$$

$\Rightarrow$ u, v are fu^n depe.

3) $u = y+z$, $v = x+2z^2$, $w = x + 4yz - 2y^2$

$\longrightarrow$ $w = v - 2u^2$

1) Suppose $u = f_1(x,y)$, $v = f_2(x,y)$

u, v are functionally dependent if

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

2) Suppose $u = f_1(x,y,z)$, $v = f_2(x,y,z)$

$w = f_3(x,y,z)$

$$u, v, w \longrightarrow x, y, z.$$

u, v, w are functionally depended if

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

3) Suppose $u = f_1(x,y,z)$, $v = f_2(x,y,z)$.

$$u, v \longrightarrow x, y, z.$$

u, v are functionally dependent if

$$\frac{\partial(u,v)}{\partial(x,y)} = 0, \quad \frac{\partial(u,v)}{\partial(y,z)} = 0, \quad \frac{\partial(u,v)}{\partial(x,z)} = 0$$

Que Check whether following functions are dependent. If so find relⁿ betⁿ them.

1) $u = x+y$, $v = 3xy$.

$\rightarrow$ u, v are funⁿ dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3y & 3x \end{vmatrix}$$

$$= 3x - 3y \neq 0$$

$\Rightarrow$ u, v are not funⁿ dependent.

2) $u = x^2+y^2$ $v = 10xy$.

$\rightarrow$ u, v are funⁿ dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 10y & 10x \end{vmatrix}$$

$$= 20x^2 - 20y^2 \neq 0$$

$\Rightarrow$ u, v are not funⁿ dependent.

3) $u = x+y$, $v = \sin(x+y)$.

$\rightarrow$ u, v are funⁿ dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \cos(x+y) & \cos(x+y) \end{vmatrix}$$

$$= \cos(x+y) - \cos(x+y) = 0$$

$\Rightarrow$ u, v are fun dependent

<u>Relⁿ</u> Given $u = x+y$, $v = \sin(x+y)$.

$$\Rightarrow v = \sin(x+y)$$

$$\Rightarrow \boxed{v = \sin u}$$

4) $u = \dfrac{x-y}{1+xy}$, $v = \tan^{-1}x - \tan^{-1}y$

$\longrightarrow$ u, v are fun dependent if $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

Given $u = \dfrac{x-y}{1+xy}$ diff wrt x

$$\frac{\partial u}{\partial x} = \frac{(1+xy)[1] - (x-y)[y]}{(1+xy)^2} = \frac{1+xy-xy+y^2}{(1+xy)^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1+y^2}{(1+xy)^2}}$$

5. $u = \dfrac{x-y}{1+xy}$ diffⁿ wrt y

$$\frac{\partial u}{\partial y} = \frac{(1+xy)[-1] - (x-y)[x]}{(1+xy)^2}$$

$$= \frac{-1-xy-x^2+xy}{(1+xy)^2} = \boxed{-\frac{1+x^2}{(1+xy)^2}}$$

Also $V = \tan^{-1}x - \tan^{-1}y$

$$\frac{\partial V}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial V}{\partial y} = -\frac{1}{1+y^2}$$

Now
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ V_x & V_y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1+y^2}{(1+xy)^2} & -\dfrac{1+x^2}{(1+xy)^2} \\ \dfrac{1}{1+x^2} & -\dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{-1}{(1+xy)^2} + \frac{1}{(1+xy)^2} = 0$$

$\Rightarrow$ u, v are Fuⁿ depended

Relⁿ $\quad u = \dfrac{x-y}{1+xy}, \quad v = \tan^{-1}x - \tan^{-1}y$

$$V = \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}u$$

$\boxed{V = \tan^{-1}u}$ or $\boxed{u = \tan V}$

5) $u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

→ Let $\alpha = \sin^{-1}x$ $\quad\quad$ $\beta = \sin^{-1}y$

$\Rightarrow x = \sin\alpha$ $\quad\quad$ $\Rightarrow y = \sin\beta$

Given $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$= \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$

$= \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \sin(\alpha+\beta)$

$= \sin(\sin^{-1}x + \sin^{-1}y)$

$$\boxed{v = \sin u}$$

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5) $u = \sin x + \sin y, \quad V = \sin(x+y)$

$\longrightarrow$ u, v are Fun depedent if $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ V_x & V_y \end{vmatrix} = \begin{vmatrix} \cos x & \cos y \\ \cos(x+y) & \cos(x+y) \end{vmatrix}$$

$$= \cos(x+y) \begin{vmatrix} \cos x & \cos y \\ 1 & 1 \end{vmatrix}$$

$$= \cos(x+y)\left[\cos x - \cos y\right]$$

$\neq 0$

$\Rightarrow$ u & V are not Fun dependent.

6) $u = x + y + z, \quad V = x^2 + y^2 + z^2, \quad w = xy + yz + zx$

$\longrightarrow u^2 = (x+y+z)^2$

$x \quad y \quad z$

$$= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

$$= (x^2 + y^2 + z^2) + 2(xy + xz + yz)$$

$$\boxed{u^2 = V + 2w}$$

7) $u = x+y+z$, $V = x-y+z$, $w = x^2+y^2+z^2+2xz$

$\longrightarrow$ $u^2 = (x+y+z)^2$

$= x^2+y^2+z^2+2xy+2xz+2yz$

$V^2 = (x-y+z)^2$ $\quad x \quad -y \quad z$

$= x^2+y^2+z^2-2xy+2xz-2yz$

$u^2+V^2 = 2x^2+2y^2+2z^2+4xz$

$= 2\left[x^2+y^2+z^2+2xz\right]$

$\boxed{u^2+V^2 = 2w}$

8) $u = \underset{1}{y}+z$, $V = \underset{1}{x}+2\underset{2}{z^2}$, $w = \underset{1}{x}-4\underset{2}{yz}-2\underset{2}{y^2}$

$\longrightarrow$ $u^2 = (y+z)^2 = y^2+2yz+z^2$

LHS $= V-2u^2$

$= (x+2z^2) - 2(y^2+2yz+z^2)$

$= x+\cancel{2z^2}-2y^2-4yz-\cancel{2z^2}$

$= x-4yz-2y^2 = w \Rightarrow \boxed{V-2u^2 = w}$

9) $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$

$\longrightarrow$ $uv = \frac{x-y}{\cancel{x+y}} \times \frac{\cancel{x+y}}{x} = \frac{x-y}{x}$

$uv = \frac{x}{x} - \frac{y}{x} = 1 - \frac{y}{x}$

But $v = \frac{x+y}{x} = 1 + \frac{y}{x} \Rightarrow \frac{y}{x} = v-1$

$uv = 1 - \frac{y}{x} = 1 - (v-1)$

$\boxed{uv = 2 - v}$

10) $u = \frac{x-y}{x+y}$, $v = \frac{xy}{(x+y)^2}$

$\longrightarrow$ $u^2 = \left(\frac{x-y}{x+y}\right)^2 = \frac{x^2-2xy+y^2}{(x+y)^2}$

$u^2 + 4v = \frac{x^2-2xy+y^2}{(x+y)^2} + \frac{4xy}{(x+y)^2}$

$= \frac{x^2+2xy+y^2}{(x+y)^2} = \frac{(x+y)^2}{(x+y)^2} = 1$

$\boxed{u^2 + 4v = 1}$

Three method to find Jacobian

1) Define (explicit fun)

$$u = f_1(x,y), \quad v = f_2(x,y) \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

2) Use of $JJ' = 1$

3) Jacobian of implicit fun

$$f_1(u,v,x,y) = 0, \quad f_2(u,v,x,y) = 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}.$$

Three Applications of Jacobian

1) $$\iint_A f(x,y)\,dx\,dy = \iint_A f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta$$

$$r = \frac{\partial(x,y)}{\partial(r,\theta)}$$

2) To find partial deri of implicit fun

3) Functional Dependence

- **Error & Approxi**

Suppose $x \rightarrow$ any phy quantity

$dx \rightarrow$ Absolute error/change in x.

$\frac{dx}{x} \rightarrow$ Relative error in x

$100\frac{dx}{x} \rightarrow$ % error in x

x	dx
100	$?$

$\Rightarrow \quad ?\cdot x = 100dx$

$? = \frac{100dx}{x}$

Formulae

1) Area of Circle $A = \pi r^2$
2) Area of ellipse $A = \pi ab$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3) Volume of sph $V = \frac{4}{3}\pi r^3$
4) Volume of cylinder $V = \pi r^2 h$

i) Find % error in area of an ellipse when 1% & 2% errors are made in measuring semi-major & semi-minor axes.

→ Eqⁿ of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

- % error in semi-major axis (a) → 1%

  i.e. $100\frac{da}{a}=1$

- % error in semi-minor axis (b) → 2%

  i.e. $\frac{100\,db}{b}=2$

Area of ellipse $A=\pi ab$

taking log ~~diff~~ on both sides

$\log A=\log\pi+\log a+\log b$

diffⁿ on both sides

$$\frac{dA}{A}=0+\frac{da}{a}+\frac{db}{b}$$

multi by 100

$$100\frac{dA}{A}=100\frac{da}{a}+100\frac{db}{b}=1+2=\boxed{3}$$

⇒ % error in area of ellipse is $\underline{\underline{3}}$.

Q7) In calculating volume of RCCylin 2|4|21 (1)

error of 2% & 1% are made in measuring height & radius. Find % error in volume

→ Volume of R.C. Cylinder is

$$V = \pi r^2 h$$

Given $100\frac{dr}{r} = 1$, $100\frac{dh}{h} = 2$

Taking log

$$\log V = \log \pi + 2\log r + \log h$$

diffⁿ

$$\frac{dv}{V} = 0 + \frac{2\,dr}{r} + \frac{dh}{h}$$

mult by 100

$$100\frac{dv}{V} = 2\left(100\frac{dr}{r}\right) + 100\frac{dh}{h}$$

$$= 2(1) + 2$$

$$\boxed{100\frac{dv}{V} = 4}$$

% error in volume of R.C.C. is 4.

3) The resonant freq in a series electric ckt is $f = \frac{1}{2\pi\sqrt{LC}}$. If measurement of L & C are in error by 2% & −3% respect. Find % error in f?

$\underset{\text{(increased)}}{2\%}$ & $\underset{\text{(decreased)}}{-3\%}$

→ freq is $f = \frac{1}{2\pi\sqrt{LC}}$

$$\log f = \log\left(\frac{1}{2\pi\sqrt{LC}}\right)$$

$$= \underset{0}{\underset{\parallel}{\log 1}} - \log\left[2\pi\sqrt{LC}\right]$$

$$= - \log 2\pi - \log (LC)^{1/2}$$

$$\log f = -\log(2\pi) - \frac{1}{2}\log L - \frac{1}{2}\log C.$$

diffⁿ

$$\frac{df}{f} = 0 - \frac{1}{2}\frac{dL}{L} - \frac{1}{2}\frac{dc}{c}$$

multi by 100

$$100\frac{df}{f} = -\frac{1}{2}\left(100\frac{dL}{L}\right) - \frac{1}{2}\left(100\frac{dc}{c}\right)$$

$$= -\frac{1}{2}(2) - \frac{1}{2}(-3)$$

$$= -1 + 3/2 = 1/2$$

$$\boxed{100\frac{df}{f} = \frac{1}{2\%}}$$

4) The focal length of mirror is obtained by $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the % error in f if both u & v are in error by p %. (2)

→ Given $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$ —— (1)

diffⁿ on both sides

$$-\frac{2}{f^2}\,df = -\frac{1}{v^2}\,dv + \frac{1}{u^2}\,du$$

multi- by 100

$$-\frac{2}{f}\left(100\frac{df}{f}\right) = -\frac{1}{v}\left(100\frac{dv}{v}\right) + \frac{1}{u}\left(100\frac{du}{u}\right)$$

Given $100\frac{dv}{v} = 100\frac{du}{u} = p$

$$-\frac{2}{f}\left(100\frac{df}{f}\right) = -\frac{1}{v}(p) + \frac{1}{u}(p)$$

$$-\text{''} \text{———} = -p\left(\frac{1}{v} - \frac{1}{u}\right)$$

$$-\frac{2}{f}\left(100\frac{df}{f}\right) = -p\left(\frac{2}{f}\right)$$

$$\boxed{100\frac{df}{f} = p}$$

5) The are of ΔABC is given by $A = \frac{1}{2} bc\sin\theta$. Error of 1%, 2% & 3% are made in measuring b, c & θ respect. If correct value of θ is 30° find % error in A

→ Area is $A = \frac{1}{2} bc\sin\theta$ —— ①

Taking log

$$\log A = \log\frac{1}{2} + \log b + \log c + \log(\sin\theta)$$

diff

$$\frac{dA}{A} = 0 + \frac{db}{b} + \frac{dc}{c} + \frac{1}{\sin\theta}\underbrace{\cos\theta\, d\theta}_{\text{mult by } \theta \text{ \& divide}}$$

$$\frac{dA}{A} = \frac{db}{b} + \frac{dc}{c} + \cot\theta\left(\frac{d\theta}{\theta}\right)\cdot\theta$$

$$100\frac{dA}{A} = 100\frac{db}{b} + 100\frac{dc}{c} + \theta\cot\theta\left(100\frac{d\theta}{\theta}\right)$$

Given $100\frac{db}{b} = 1$, $100\frac{dc}{c} = 2$

$$100\frac{d\theta}{\theta} = 3$$

$$100\frac{dA}{A} = 1+2+\theta\cot\theta(3)$$

But $\theta = 30^\circ = \frac{\pi}{6}$

$$100\frac{dA}{A} = 3+3\frac{\pi}{6}\cot\left(\frac{\pi}{6}\right)$$

$$= 3+\frac{\pi}{2}\left(\sqrt{3}\right)$$

$$\boxed{100\frac{dA}{A} = 3+\frac{\sqrt{3}\pi}{2}}$$

% error in A is $3+\frac{\sqrt{3}}{2}\pi$.

6) The k.E is calculated using $T=\frac{mv^2}{2}$ & m changes from 49 to 49.5 & V changes from 1600 to 1590, find

a) Appnx. change in T.

b) % error in T.

$\longrightarrow$ k.E is $T=\frac{mv^2}{2}$

$$\log T = \log\frac{1}{2}+\log m+2\log V$$

diffⁿ

$$\frac{dT}{T} = 0+\frac{dm}{m}+\frac{2dv}{V}$$

a) <u>% error</u>

$$100\frac{dT}{T} = 100\frac{dm}{m} + 2\ 100\frac{dv}{v}$$

Given $v \Rightarrow 1600 \xrightarrow{\text{to}} 1590$

$m \Rightarrow 49 \longrightarrow 49.5$

m_0 = orig value = 49 , m_1 = final value = 49.5

V_0 = orig value = 1600 , V_1 = final value = 1590

$$dm = m_1 - m_0 = 49.5 - 49 = 0.5$$

$$dv = V_1 - V_0 = 1590 - 1600 = -10$$

$$100\frac{dT}{T} = 100\left(\frac{0.5}{49}\right) + 2 \times 100 \times \left(\frac{-10}{1600}\right)$$

$$\boxed{100\frac{dT}{T} = -0.2495} \Rightarrow \text{\% error in K.E.}$$

b) <u>Approx. change (dT)</u>

$$dT = \frac{T}{100}(-0.2495)$$

$$= \frac{mv^2}{2 \times 100}(-0.2495)$$

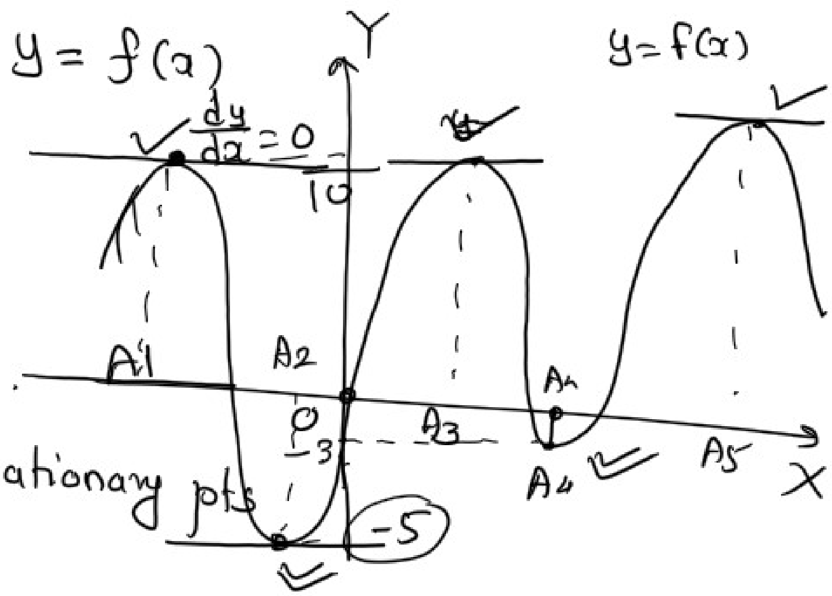
$$= \frac{49 \times (1600)^2}{200} \times (-0.2495)$$

$$\boxed{dT = -156486.4 \text{ Cal.}}$$ Approx. change in K.E.

Maxima & Minima

Aim - To find max or min of $u = f(x, y)$.

Revision



$$\frac{dy}{dx} = 0$$
$$\Downarrow$$

$x = a, b, c, \ldots$

Critical pts / stationary pts

$$\left.\frac{d^2y}{dx^2}\right|_{x=a} > 0 \rightarrow \text{minima}$$

$$< 0 \rightarrow \text{Maxima}$$

$$= 0 \rightarrow \text{~~Saddle pt~~ Can't be decid...}$$

$$\frac{d^4y}{dx^4} > 0 \rightarrow \text{mini}$$

$$< 0 \rightarrow \text{max}$$

$$= 0$$

$y \longrightarrow x$

$\left\{ \begin{array}{l} z \rightarrow x, y \\ u \longrightarrow x, y, z \end{array} \right.$

Maxima A fn $f(x, y)$ is said to have maxima at (a, b) if

$$f(x, y) \leq f(a, b) \quad \forall \; x, y$$

Minima

A Fun $F(x,y)$ is said to have minima at (c,d) if $f(x,y) > f(c,d)$ $\forall x,y$.

Critical Points / Stationary Points

The pts at which $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ are known as - - -

Saddle points

The stationary pts where max or minima doesnot exist are known as - - - .

I) General Method

To find extreme pts of $u = F(x,y)$.

1) Find First & second order deri

$$p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial y}.$$

$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}$$

2) Stationary pts - are obtained from

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

3) Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

at every critical pt.

4) If

a) $rt - s^2 > 0$ & $r < 0 \Rightarrow$ Maxima exist.

b) $rt - s^2 > 0$ & $r > 0 \Rightarrow$ Minima exist

c) $rt - s^2 < 0 \Rightarrow$ Neither max nor minima exist & such pts are saddle pt.

d) $rt - s^2 = 0 \Rightarrow$ Cannot be decided.

$$f(x,y) = x^2 + y^2 + 6x + 12$$

$$p = \frac{\partial f}{\partial x} = 2x + 6 \quad , \quad q = \frac{\partial f}{\partial y} = 2y$$

$$\swarrow x \qquad \searrow y \qquad \qquad \downarrow y$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \quad , \quad s = \frac{\partial^2 f}{\partial y \partial x} = 0 \quad , \quad t = \frac{\partial^2 f}{\partial y^2} = 2$$

<u>Stationary pts</u> Are obtained from $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$2x + 6 = 0 \quad \& \quad 2y = 0$$

$$x = -3 \quad \& \quad y = 0$$

Stationary pt is $\boxed{A(-3,0)}$

Now $r = 2$, $s = 0$, $t = 2$

$$rt - s^2 = (2)(2) - (0)^2 = 4 > 0 \quad \& \quad r = 2 < 0$$

$\Rightarrow$ Maxima exist at $A(-3,0)$ & maxi value is

$$f_{max} = x^2 + y^2 + 6x + 12 \Big|_{(-3,0)}$$

$$f_{max} = (-3)^2 + (0)^2 + 6(-3) + 12$$

$$f_{max} = 9 + 0 - 18 + 12 = \boxed{3}$$

ii) Find extreme values of the Fn $3x^2-y^2+x^3$ (3)

$\rightarrow$ Here $F(x,y) = x^3+3x^2-y^2$

$p = \frac{\partial f}{\partial x} = 3x^2+6x$, $q = \frac{\partial f}{\partial y} = -2y$

$r = \frac{\partial^2 f}{\partial x^2} = 6x+6$, $s = \frac{\partial^2 f}{\partial y \partial x} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = -2$

Stationary pts Are obtained from

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$3x^2+6x = 0$, $-2y = 0$

$3x(x+2) = 0$, $y = 0$

$x = 0, -2$

Stationary pts $A(0,0)$ & $B(-2,0)$.

- At $A(0,0)$ ie. $x=0, y=0$

$r = 6x+6 = 6(0)+6 = 6$

$s = 0$, $t = -2$

$rt - s^2 = (6)(-2)-(0)^2 = -12 < 0$

Neither max nor min exist at $A(0,0)$

& it is known as Saddle pt.

At $B(-2,0)$ i.e. $x=-2,\ y=0$

$r = \cancel{3x^2+3y^2-30x+72}\ 6x+6 = -6$ $\quad 6xy-30y=0$

$s = 0\ ,\quad t = -2$

$rt - s^2 = (-6)(-2) - (0) = 12 > 0\ \&\ r = -6 < 0$

$\Rightarrow$ Maxima exist at $B(-2,0)$

Maximum value is

$$f_{max} = x^3 + 3x^2 - y^2 \Big|_{(-2,0)}$$

$$= (-2)^3 + 3(-2)^2 - (0)^2$$

$$= -8 + 12 - 0$$

$$\boxed{f_{max} = 4}$$

3) Find extreme values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

$\rightarrow$ Given $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

$$p = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$q = \frac{\partial f}{\partial y} = 6xy - 30y$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30 \quad x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y \leftarrow x \quad y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30. \quad y$$

Stationary pts $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$ & $6xy - 30y = 0$

$x^2 + y^2 - 10x + 24 = 0$ & $6y(x-5) = 0$

$\Downarrow$

$y = 0$ & $x = 5$

$(5,0)$

- when $y=0$

$x^2 - 10x + 24 = 0$

$(x-4)(x-6) = 0$

$x = 4, 6 \Rightarrow \boxed{A(4,0), \ B(6,0)}$

- when $x = 5$

$25 + y^2 - 50 + 24 = 0$

$y^2 - 1 = 0$

$y^2 = 1 \Rightarrow y = \pm 1$

$\boxed{C(5,-1) \ \& \ D(5,1)}$

$x + y = 5$

$x - y = 3$

$x = 3, \ y = 0$

Here $r = 6x-30$, $s = 6y$, $t = 6x-30$

- at A (4, 0) ie. $x = 4, y = 0$

$r = 6x-30 = 6(4)-30 = -6$

$s = 6y = 6(0) = 0$, $t = 6x-30 = -6$

$rt - s^2 = (-6)(-6) - 0 = 36 > 0$ & $r = -6 < 0$

$\Rightarrow$ Maxima exist at A (4, 0).

$f_{max} = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \Big|_{(4,0)}$

$f_{max} = 64 + 0 - 240 - 0 + 288$

$\boxed{f_{max} = 112}$

- At B (6, 0) ie. $x = 6, y = 0$

$r = t = 6x - 30 = 6(6) - 30 = 6$

$s = 6y = 6(0) = 0$

$rt - s^2 = (6)(6) - (0) = 36 > 0$ & $r = 6 > 0$

$\Rightarrow$ Minima exist at B (6, 0)

$f_{min} = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \Big|_{(6,0)}$

$= 216 + 0 - 540 - 0 + 432$

$\boxed{f_{min} = 108}$

• At $C(5,-1)$ ie. $x=5,\ y=-1$

$r = t = 6x-30 = 6(5)-30 = 0$

$s = 6y = 6(-1) = -6$

$rt - s^2 = (0) - (-6)^2 = -36 < 0$

$\Rightarrow$ $C(5,-1)$ is saddle pt

At $D(5,1)$ ie. $x=5,\ y=1$

$r = t = 6x-30 = 6(5)-30 = 0$

$s = 6y = 6(1) = 6$

$rt - s^2 = (0) - (6)^2 = -36 < 0$

$\Rightarrow$ $D(5,1)$ is saddle pt.

Result

St. pt	Condition	Conclusion	Remark
$A(4,0)$	$rt-s^2>0,\ r<0$	Maxima exist	$f_{max} = 112$
$B(6,0)$	$rt-s^2>0,\ r>0$	Minima exist	$f_{min} = 108$
$C(5,-1)$	$rt-s^2<0$	Saddle pt	—
$D(5,1)$	$rt-s^2<0$	Saddle pt	—

Homework

1) Show that minimium value of

$xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$

2) Find extreme values of fu^n

$x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Lagrange's Method of Undetermined Multiplier

This method is used to find stationary pts of a fu^n having three or more independent variables. Drawback is that, we cannot decide whether maxima or minima exist.

Procedure

To optimize $u = f(x, y, z)$ under the cond^n $\phi(x, y, z) = 0$

Construct a fu^n $F = u + \lambda\phi$

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

i) Find max. value of $u = x^2 y^3 z^4$
such that $2x + 3y + 4z = a$

$\longrightarrow$ To maximize $u = x^2 y^3 z^4$ under the
cond^n $2x + 3y + 4z = a$ — (1)

ie. $\phi = 2x + 3y + 4z - a = 0$

Construct a fu^n $F = u + \lambda \phi$

$$F = \{ x^2 y^3 z^4 \} + \lambda \{ 2x + 3y + 4z - a \}.$$

$$\frac{\partial F}{\partial x} = 2x y^3 z^4 + 2\lambda = 0 \Rightarrow \lambda = -x y^3 z^4$$

$$\frac{\partial F}{\partial y} = 3x^2 y^2 z^4 + 3\lambda = 0 \Rightarrow \lambda = -x^2 y^2 z^4$$

$$\frac{\partial F}{\partial z} = 4x^2 y^3 z^3 + 4\lambda = 0 \Rightarrow \lambda = -x^2 y^3 z^3$$

Equating λ $\quad \lambda = \lambda = \lambda$

$$-x y^3 z^4 = -x^2 y^2 z^4 = -x^2 y^3 z^3$$

dividing by $x^2 y^3 z^4$

$$+\frac{x y^3 z^4}{x^2 y^3 z^4} = \frac{x^2 y^2 z^4}{x^2 y^3 z^4} = \frac{x^2 y^3 z^{\cancel{4}\,3}}{x^2 y^3 z^4}$$

$$\frac{1}{x} = \frac{1}{y} = \frac{1}{z} = \frac{1}{k} \text{ (say)}.$$

$$x = y = z = k$$

But $\quad 2x + 3y + 4z = a$

$$2k + 3k + 4k = a$$

$$9k = a$$

$$\boxed{k = a/9}$$

Thus $\quad x = y = z = k = \frac{a}{9}$

St pt is $\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right)$.

Maximum value if f ~~...~~ $U_{max} = x^2 y^3 z^4$

$$= \left(\frac{a}{9}\right)^2 \left(\frac{a}{9}\right)^3 \left(\frac{a}{9}\right)^4$$

~~2)~~

$$\boxed{U_{max} = \left(\frac{a}{9}\right)^9}$$

2) Find stationary pt for $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$

where $x + y + z = 1$

$\longrightarrow$ To optimize $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$

under the condⁿ $\cancel{\phi}$ $x + y + z = \cancel{\phi}$ — (1)

ie. $\phi = x + y + z - 1$

3) Construct a fun

$$F = u + \lambda \phi$$

$$F = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} + \lambda (x + y + z - 1)$$

$$\frac{\partial F}{\partial x} = -\frac{2a^3}{x^3} + \lambda = 0 \Rightarrow \lambda = \frac{2a^3}{x^3}$$

$$\frac{\partial F}{\partial y} = -\frac{2b^3}{y^3} + \lambda = 0 \Rightarrow \lambda = \frac{2b^3}{y^3}$$

$$\frac{\partial F}{\partial z} = -\frac{2c^3}{z^3} + \lambda = 0 \Rightarrow \lambda = \frac{2c^3}{z^3}$$

Equating λ $\quad \lambda = \lambda = \lambda$

$$\frac{2a^3}{x^3} = \frac{2b^3}{y^3} = \frac{2c^3}{z^3}$$

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{1}{k} \text{ (say)}$$

$$\Rightarrow \frac{a}{x} = \frac{1}{k}, \quad \frac{b}{y} = \frac{1}{k}, \quad \frac{c}{z} = \frac{1}{k}$$

$$\boxed{x = ak, \quad y = bk, \quad z = ck}$$

Since $x + y + z = 1$

$ak + bk + ck = 1 \Rightarrow$ $\boxed{k = \frac{1}{a+b+c}}$

$\Rightarrow$ stationary pt is

$$\boxed{x = ak = \frac{a}{a+b+c}, \quad y = bk = \frac{b}{a+b+c}, \quad z = ck = \frac{c}{a+b+c}}$$

3) If the dimensions of ΔABC are varied show that max value of $\cos A \cdot \cos B \cdot \cos C$ is obtained when Δ is equilateral.

$\longrightarrow$ To maximize $u = \cos A \cos B \cos C$ under the condⁿ $A + B + C = \pi$ —— ①

ie. $\phi = A + B + C - \pi = 0$

Construct a fuⁿ $F = u + \lambda \phi$

$$F = \cos A \cos B \cos C + \lambda (A + B + C - \pi).$$

$$\frac{\partial F}{\partial A} = -\sin A \cos B \cos C + \lambda = 0 \Rightarrow \lambda = \sin A \cos B \cos C$$

$$\frac{\partial F}{\partial B} = -\cos A \sin B \cos C + \lambda = 0 \Rightarrow \lambda = \cos A \sin B \cos C$$

$$\frac{\partial F}{\partial C} = -\cos A \cos B \sin C + \lambda = 0 \Rightarrow \lambda = \cos A \cos B \sin C$$

Equating λ $\quad \lambda = \lambda = \lambda$

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$$

dividing by $\cos A \cos B \cos C$

$$\tan A = \tan B = \tan C$$

$$\Rightarrow A = B = C$$

But $A + B + C = \pi \Rightarrow A + A + A = \pi \Rightarrow A = \frac{\pi}{3}$

$$\boxed{A = B = C = \pi/3}$$

ΔABC is equilateral Δ.

4) Divide 120 into three parts such that sum of their products taken two at a time shall be maximum.

→ Suppose three parts of 120 are x, y, z

so $x+y+z = 120$ — ①

To maximize $u = xy + yz + zx$.

Under the condⁿ $\phi = x+y+z-120 = 0$

Construct a fuⁿ $F = u + \lambda\phi$

$$F = xy + yz + zx + \lambda(x+y+z-120).$$

$$\frac{\partial F}{\partial x} = y+z+\lambda = 0 \Rightarrow \lambda = -(y+z)$$

$$\frac{\partial F}{\partial y} = x+z+\lambda = 0 \Rightarrow \lambda = -(x+z)$$

$$\frac{\partial F}{\partial z} = y+x+\lambda = 0 \Rightarrow \lambda = -(x+y)$$

Equating λ $\lambda = \lambda = \lambda$

$$-(y+z) = -(x+z) = -(x+y)$$

$$y+z = x+z = x+y = k.$$

$$y+z = k,\ x+y = k,\ x+z = k$$

$$\boxed{x = y = z = k/2}$$

5) B~~A~~ spac. $x+y+z = 120$

$$\frac{k}{2}+\frac{k}{2}+\frac{k}{2} = 120$$

$$3k = 240$$

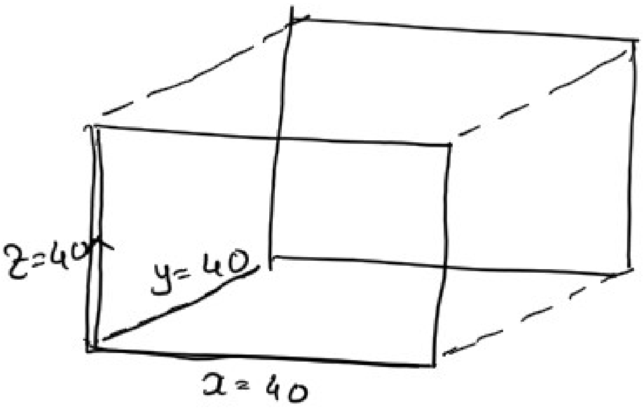
$$\boxed{k = 80}$$

$$\boxed{x = y = z = \frac{k}{2} = 40}$$

Explaination.

Cube with sides

$x = y = z = 40$



$$U = xy + yz + z\cancel{x}$$

$$= \frac{1}{2}\left[2xy + 2yz + 2xz\right]$$

$$= \frac{1}{2}\left[\text{surface area}\right]$$

~~when~~ cube will provide maxi surface area such that $x+y+z = 120$.

5) A space probe in the shape of ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere & its surface begins to heat. After 1 hr, the temp at any pt (x,y,z) on surface of probe is $T(x,y,z) = 8x^2+4yz-16z+600$. Find hottest pt on surface of probe.

$\longrightarrow$ To maxi $T = 8x^2+4yz-16z+600$

under the condn that pt (x,y,z) lies on surface of probe $4x^2+y^2+4z^2=16$ — ①

ie. $\phi = 4x^2+y^2+4z^2-16 = 0$

Construct a fun $F = T+\lambda\phi$

$$F = (8x^2+4yz-16z+600)+\lambda(4x^2+y^2+4z^2-16)$$

$$\frac{\partial F}{\partial x} = 16x+8x\lambda = 0 \Rightarrow 8x\lambda = -16x$$

$$\boxed{\lambda = -2}$$

$$\frac{\partial F}{\partial y} = 4z+2y\lambda = 0 \quad ②$$

$$\frac{\partial F}{\partial z} = 4y-16+8z\lambda = 0 \quad ③$$

Subst $\lambda=-2$ in ② & ③

$4z+2y(-2)=0$

$4z-4y=0 \Rightarrow \boxed{y=z}$

③ $\Rightarrow$ $4y - 16 + 8z\lambda = 0$

$4y - 16 + 8(y)(-2) = 0$

$4y - 16y = 16$

$-12y = 16$

$\boxed{y = -4/3}$

But $\boxed{y = z = -4/3}$

But $4x^2 + y^2 + 4z^2 = 16$

$4x^2 + \left(\frac{-4}{3}\right)^2 + 4\left(\frac{-4}{3}\right)^2 = 16$

$4x^2 = 16 - \frac{64}{9} - \frac{16}{9}$

$4x^2 = \frac{144 - 64 - 16}{9} = \frac{64}{9}$

$x^2 = \frac{16}{9} \Rightarrow \boxed{x = \pm \frac{4}{3}}$

Hottest pt on surface of probe is

$\left(\pm \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$